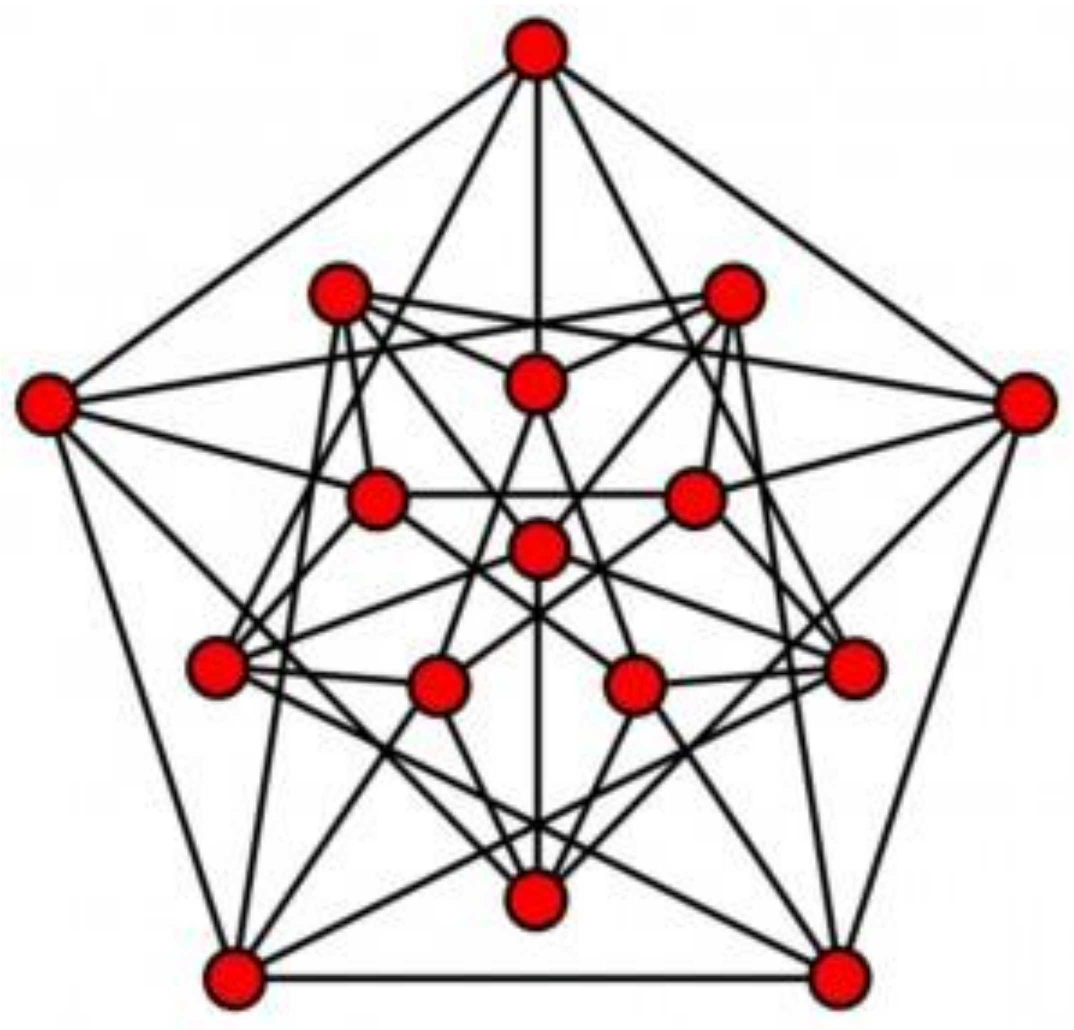
GRAPH THEORY

10th Grade Project Week 23-24

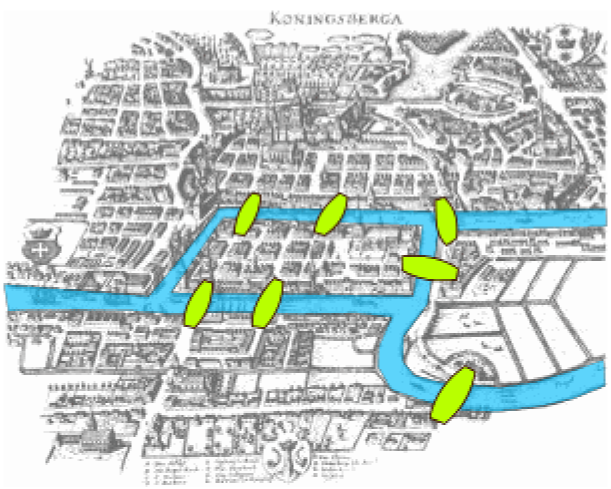


***Notes, examples, explorations, and problems taken from a variety of sources and courses. See Ms. Maslow for bibliography.**

WHAT IS GRAPH THEORY?

Graph Theory was formalized by Leonhard Euler, when he published a solution to the problem of the seven bridges of Konigsberg.

The Pregel River runs through the city of Konigsberg, Prussia. In the middle of the city, there are seven bridges that connect the shore and two islands, Kneiphof and Lomse, which are in the middle of the river, as shown below. As the river flowed around the islands, it divided the city into four distinct regions. The seven bridges were called Blacksmith’s bridge, Connecting Bridge, Green Bridge, Merchant’s Bridge, Wooden Bridge, High Bridge, and Honey Bridge. According to lore, the citizens of Königsberg used to spend Sunday afternoons walking around their beautiful city. While walking, the people of the city decided to create a game for themselves, their goal being to devise a way in which they could walk around the city, crossing each of the seven bridges only once. Even though none of the citizens of Königsberg could invent a route that would allow them to cross each of the bridges only once, still they could not prove that it was impossible. Lucky for them, Königsberg was not too far from St. Petersburg, home of the famous mathematician Leonard Euler.



In 1736 Euler explained why: he showed that such a walk didn't exist.

Euler's solution is surprisingly simple — once you look at the problem in the right way. The trick is to get rid of all unnecessary information. It doesn't matter what path the walk takes on the various land masses. It doesn't matter what shape the land masses are, or what shape the river is, or what shape the bridges are. So you might as well represent each land mass by a dot and a bridge by a line. You don't have to be geographically accurate at all: as long as you don't disturb the connectivity of the dots, which is connected to which, you can distort your picture in any way you like without changing the problem.

Once you have represented the problem in this way, its features are much easier to see. After playing around with it for a while you might notice the following: when you arrive at a dot via a line (that is, you enter a land mass via the bridge), then unless it is the final dot at which your walk ends, you need to leave it again, by a different line, as those are the rules of the game. That is, any dot that is not the starting and end-point of your walk needs to have an even number of lines coming out of it: for every line along which you enter there has to be one to leave.

For a walk that crosses every line exactly once to be possible, at most two dots can have an odd number of lines coming out of them. In fact there have to be either two odd dots or none at all. In the former case the two correspond to the starting and end points of the walk and in the latter the starting and end points are the same on any walk. In the Königsberg problem, however, all dots have an odd number of lines coming out of them, so a walk that crosses every bridge is impossible.

Euler's solution also laid the foundation for an area of maths that couldn't be more relevant to modern life: graph theory (or network theory). A graph (also known as a network) is a collection of vertices (nodes) connected up by edges (links). Networks are absolutely everywhere. We live in social networks, travel along road and rail networks, and rely on telephone networks and utility networks that deliver power or water. Our computers and phones are hooked up to that vast network called the internet; rivalled in complexity only by the network of neurons in our brains, which enables us to produce all these complex structures in the first place.

SECTION I: INTRODUCTION

BASIC DEFINITIONS:

Graph: A collection of vertices, some of which are connected by edges. More precisely, a pair of sets V and E where V is a set of vertices and E is a set of 2-element subsets of V (edges).

Vertex: Represented with a point

Edge: Represented with a curve

Order: The number of vertices on a graph

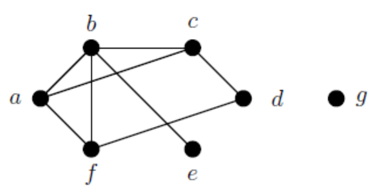
Size: The number of edges on a graph

Adjacent: Two vertices are adjacent if they are connected by an edge. Two edges are adjacent if they share a vertex.

Degree: The degree of a vertex is the number of edges connected to it.

Connected: A graph is connected if there is a path from any vertex to any other vertex.

Example:



The vertex set of the graph is $V(G) = \{a, b, c, d, e, f, g\}$
$V(G)$ has order 7.

The edge set of the graph is $E(G) = \{ab, ac, af, bf, be, bc, cd, df\}$
$E(G)$ has size 8.

a and c are adjacent.

a and e are not adjacent.

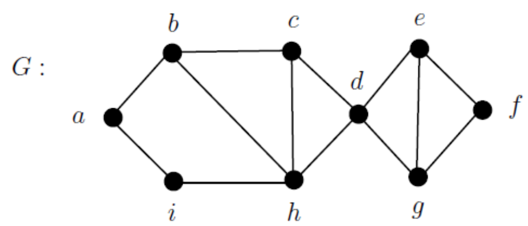
The degree of vertex c is 3.

The degree of vertex e is 1.

G is not connected, as there is not a path from g to any other vertex.

PROBLEM SET 1A

1. Answer the questions below using the graph G given here.



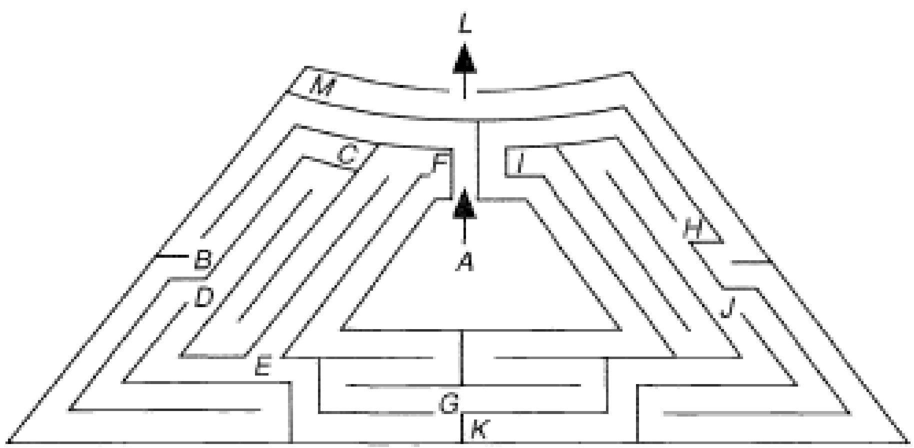
   a. List the vertex set and edge set of G.
   b. List the degrees of the vertices of G.
   c. Is G connected?
   d. Are a and c adjacent?

2. Draw the following graph

$$V = \{a, b, c, d, e, f, g\}$$
$$E = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, g\}, \{d, e\}, \{e, f\}, \{f, g\}\}$$

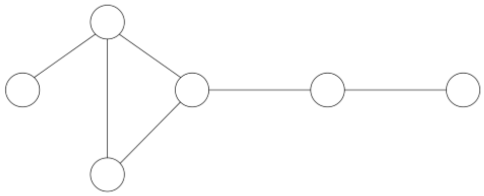
3. Create a graph with 6 vertices where two of the vertices have degree 4 and 4 of the vertices have degree 5.

4. Draw a graph with vertices A…M that shows the various routes one can take when tracing the Hampton Court maze in the figure below.



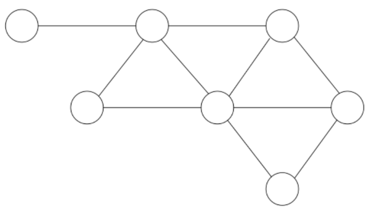
5. Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs eat snails, spiders, and insects. Draw a graph of the predatory behavior. (It might be preferable to use arrows instead of just edges in the graph, to show the direction of predation)

6. Here is a blank graph showing a group of friends. Can you work out who is who using the clues below?



Alan has 3 friends, Barney, Charlie, and Daniel.
Barney and Ed are both friends with Charlie.
Ed is Frank's only friend

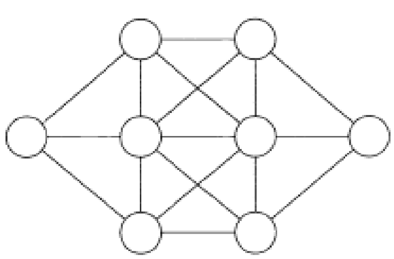
7. Here is a second network of friends. Again, use the clues below to figure out who's who.



Bella and Ciara are friends
Emily and Ciara are not friends
Bella is Fiona's only friend
Anna has more friends than anyone else
Daphne has three friends
Gill and Daphne are not friends
Emily has two friends

8. Once you've solved the two puzzles, answer the following questions:
Did each problem have a unique solution?
Were there any clues you didn't need to use?
If you label each vertex with the number of friends the person has, and add together all the numbers, what can you say about the answer? Can you explain why?
Can you design a puzzle with five friends with a unique solution?

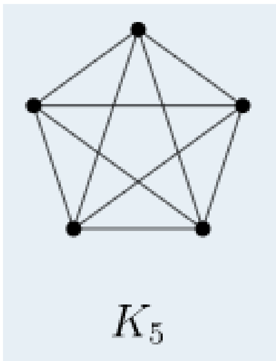
9. Place the letters A, B, C, D, E, F, G, H into the eight circles in the figures below such that no letter is adjacent to a letter next to it in the alphabet.



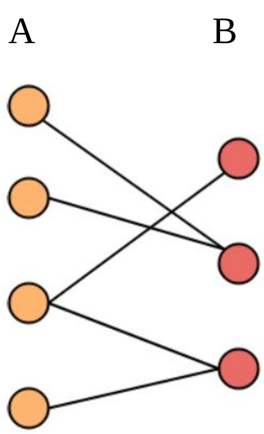
DEFINITIONS: TYPES OF GRAPHS

Complete Graph: A graph in which each vertex connects to every other vertex

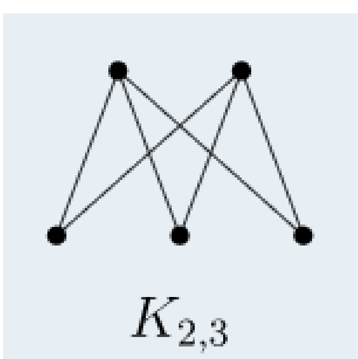
K_n = The complete graph with n vertices



Bipartite Graph: A graph where the vertices can be divided into two sets, A and B, where each vertex of A can only connect to vertices of B and vice versa. (no vertex in A connects to another vertex in A.)

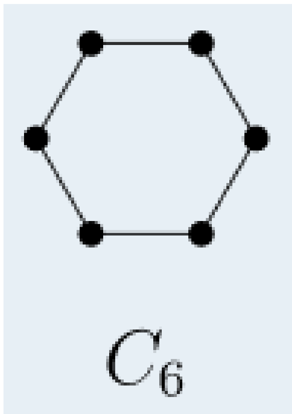


$K_{m,n}$ = The complete bipartite graph with sets of m and n vertices.

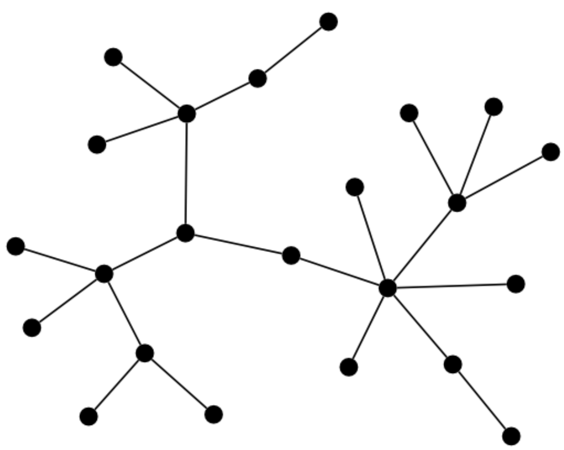


Cyclic Graph: A graph that is one simply cycle

C_n = The cycle on n vertices



Tree: A connected graph without any cycles



INVESTIGATION: COMPLETE GRAPHS

Write your work, answers, and observations in your notebook

Make the complete graphs for K_2, K_3, K_4, K_5, K_6. (remember that K_5 is the example in the definition of a complete graph).

What is the degree of each vertex? (i.e. how many edges does each vertex have?) How does this number relate to n (the n in K_n)?

How many total edges does each complete graph have? What is the relationship between that number and n?

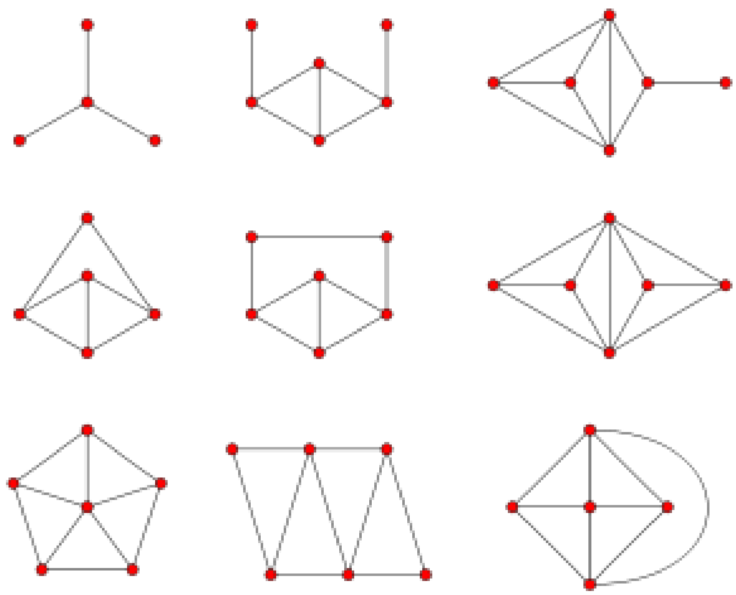
(Hint: Make a chart with three columns: one for n, one for the degree of each vertex, and one for the total edges)

Explain whey these results make sense for a complete graph.

PROBLEM SET 1B

(Your results and methods from the investigation on the previous page can be used in all of these problems, though you could also think through them without it.)

1. For the graphs below,
    a. Count the total number of edges
    b. Find the sum of the degrees of all the vertices of the following graphs.
    c. Can you see a pattern between the values for parts a and b? Explain why this makes sense.



2. For each of the following, try to give two different graphs with the given properties, or explain why doing so is impossible.

    a. 8 vertices all of degree 2.
    b. 5 vertices all of degree 4.
    c. 5 vertices all of degree 3.

3. At a recent math seminar, 9 mathematicians greeted each other by shaking hands. Is it possible that each mathematician shook hands with exactly 7 people at the seminar? If not, why not? Can you model this as a graph?

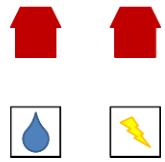
4. Among a group of 5 people, is it possible for everyone to be friends with exactly 2 of the people in the group? What about 3 of the people in the group? If possible, show with a graph. If not, why not?

SECTION II: PLANAR GRAPHS

INVESTIGATION: UTILITIES

Write your work, answers, and observations in your notebook

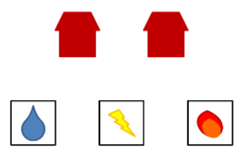
In the picture below, there are two houses that need to be connected to water and electricity.



The pipes/wires must not cross each other, and you can't go through one house to get to the other house. Can you find a way to connect them?
What if there were three houses? Or four? Or...

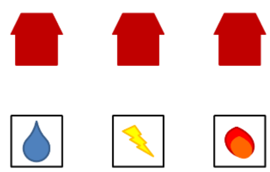
How many houses is it possible to connect to the two utilities, without requiring two lines to cross?

The Gas company wish to connect the houses to the Gas supply as well.
Here's a diagram showing the first two houses again:



Can you find a way to connect them to water, electricity and gas without any lines crossing?

What if there were three houses?

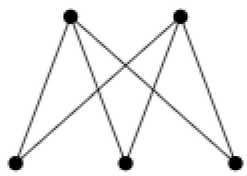


Find a way to connect three houses to three utilities, or come up with a convincing explanation why it can't be done!

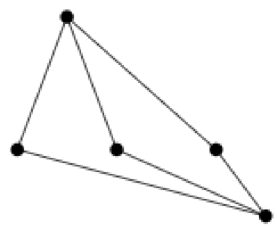
DEFINITIONS: PLANAR GRAPHS

Planar Graphs: a graph that can be drawn in the plane without edges crossing

Notice that the definition of planar includes the phrase “can be drawn.” This means that even if a graph does not look like it is planar, it still might be. Perhaps you can redraw it in a way in which no edges cross. For example, this is a planar graph:



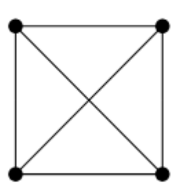
That is because we can redraw it like this:

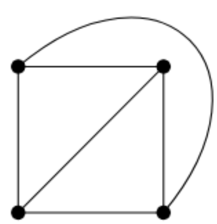


The graphs are the same, so if one is planar, the other must be too. However, the original drawing of the graph was not a ***planar representation*** of the graph.

When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a ***face***. The graph above has 3 faces, (yes, we *do* include the “outside” region as a face). The number of faces does not change no matter how you draw the graph (as long as you do so without the edges crossing), so it makes sense to ascribe the number of faces as a property of the planar graph.

WARNING: you can only count faces when the graph is drawn in a planar way. For example, consider these two representations of the same graph:

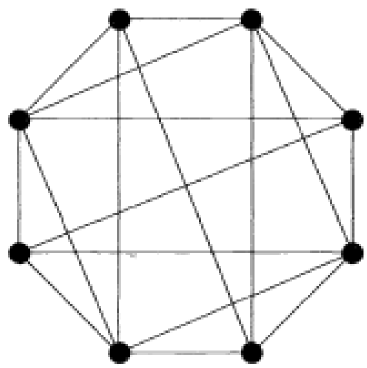


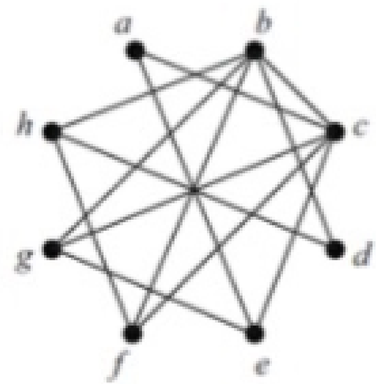


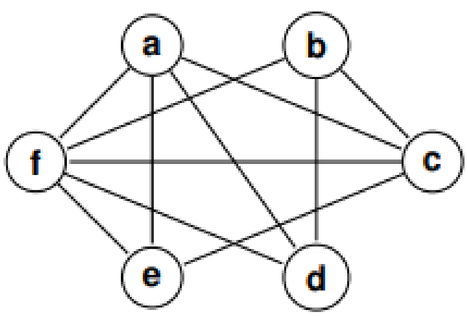
If you try to count faces using the graph on the left, you might say there are 5 faces (including the outside). But drawing the graph with a planar representation shows that in fact there are only 4 faces.

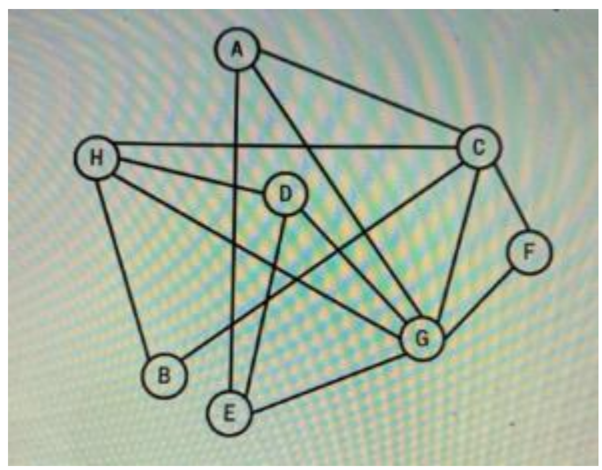
PROBLEM SET 2A

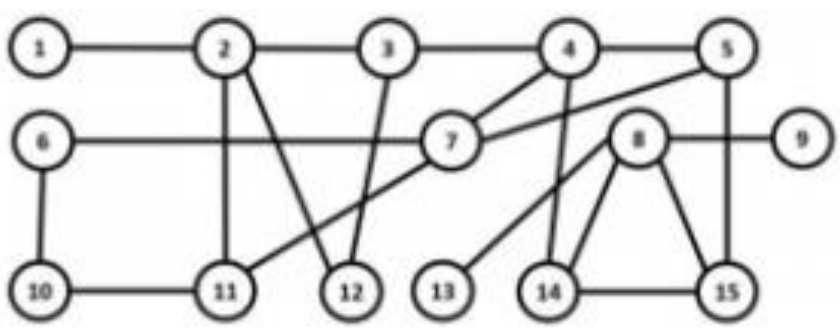
1. Show how the graphs below can be drawn in the plane without any edges crossing.











INVESTIGATION: EULER'S FORMULA

Write your work, answers, and observations in your notebook.

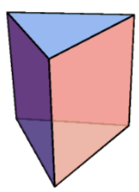
Note: This investigation is rather long. Pay close attention to all the questions asked!

A polyhedron is the three dimensional version of a polygon. Polyhedra seem to have been forgotten by mathematicians for over a century until Euler developed an interest in them. In 1750, Euler writes in a letter to a friend,

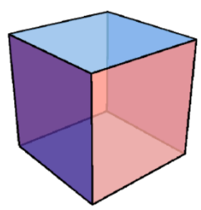
"*Recently it occurred to me to determine the general properties of solids bounded by plane faces* (polyhedra), *because there is no doubt that general theorems should be found for them, just as for plane rectilinear figures* (polygons)."

He says that he should consider *all* the bounds of a polyhedron, namely its faces, its edges, and its vertices. He was curious the connection between the three and found a simple and elegant pattern connecting them for any *convex* polyhedra. (A convex polyhedron is one where each of the vertices point out of the shape, and not into the shape.)

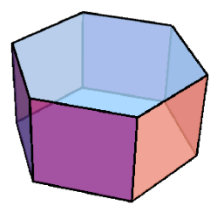
For each of the convex polyhedral below, count the number of vertices, faces, and edges, it has. (It will probably help you to make a chart to organize your data.) Can you find an algebraic relationship between the three numbers? (i.e. can you make an equation connecting them?



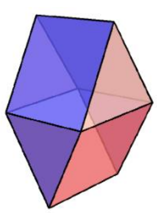
Triangular prism



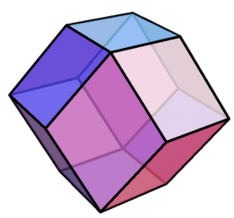
Cube



Hexagonal prism

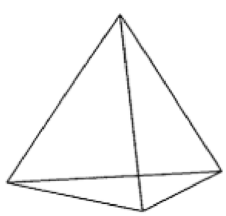


Gyrobifastigium

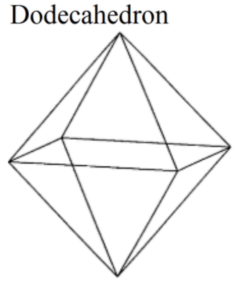


Rhombic Dodecahedron

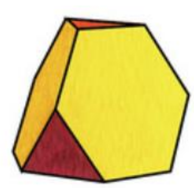
truncated **cube**



tetrahedron



octahedron



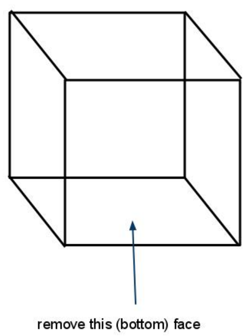
truncated **tetra**hedron

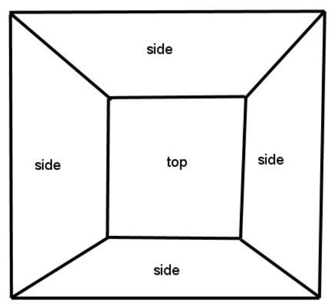
The equation connecting the three values is called *Euler's Formula.*

Interestingly, though Euler essentially invented graph theory with the Konigsberg Bridge Problem, he did not see the connection between the polyhedra problem to graphs.
Do you???

In fact, every convex polyhedral can be 'flattened' into what looks like a planar graph.

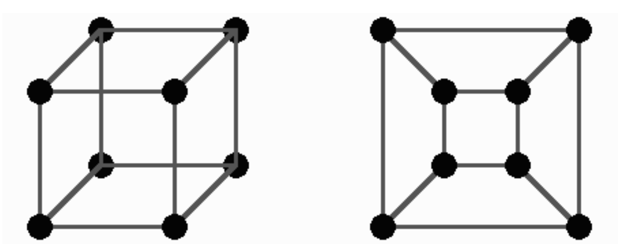
For example, if you take the cube, stretch one of its faces, and smush the rest of the cube down, you get





flattened cube

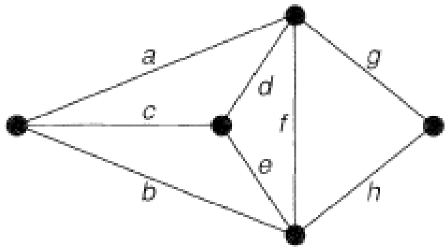
Or, to look more like a graph,

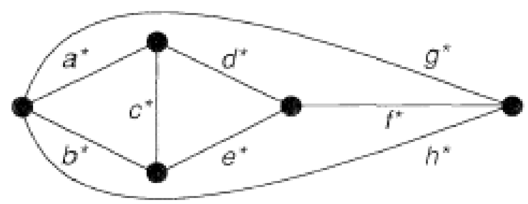


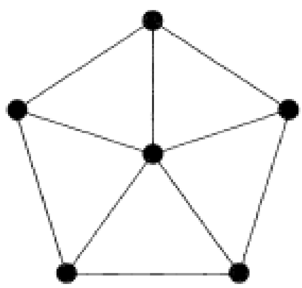
Try to do the same with a few of the polyhedra from the previous page.

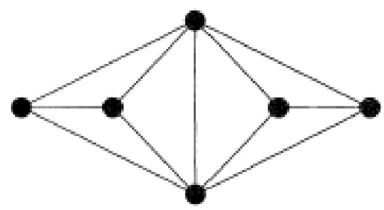
If any convex polyhedral can be made into a planar graph, does Euler's formula apply to any planar graph? Check the following graphs. Feel free to create a few more examples if helpful.

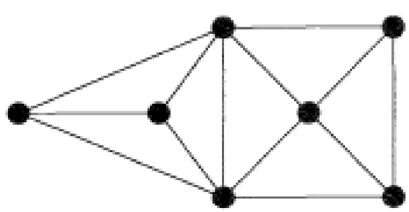
(Remember that with planar graphs we include the 'outside' when counting faces. Does that make more sense now?)

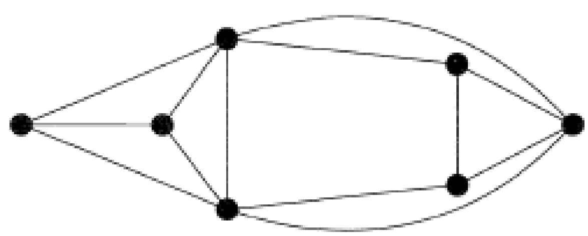












If all convex polyhedra can be flattened into planar graphs, then should it be the case that all planar graphs can be stretched back into space to be convex polyhedral? Try for a few of the examples above. What conclusion do you come to?

PROBLEM SET 2B

(Euler's formula could be used to help you think through many of these problems, though you could also think through them without it.)

1. Your friend claims that he has constructed a convex polyhedron out of 2 triangles, 2 squares, 6 pentagons and 5 octagons. Prove that your friend is lying. (Hint: each vertex of a convex polyhedron must border at least three faces.)

2. Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

3. The graph G has 6 vertices with degrees 2, 2, 3, 4, 4, 5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

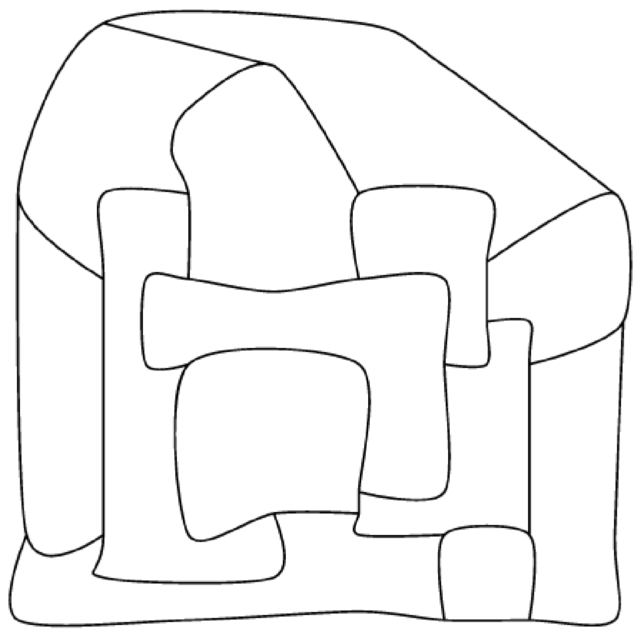
4. I'm thinking of a polyhedron containing 12 faces. Seven are triangles and four are quadrilaterals. The polyhedron has 11 vertices including those around the mystery face. How many sides does the last face have?

SECTION III: COLORING

INVESTIGATION: MAPS

Write your work, answers, and observations in your notebook.

Mapmakers in the land of Euleria have drawn the borders of the various dukedoms of the land. To make the map pretty, they wish to color each region. Adjacent regions must be colored differently, but it is perfectly fine to color two distant regions with the same color. What is the fewest colors the mapmakers can use and still accomplish this task?



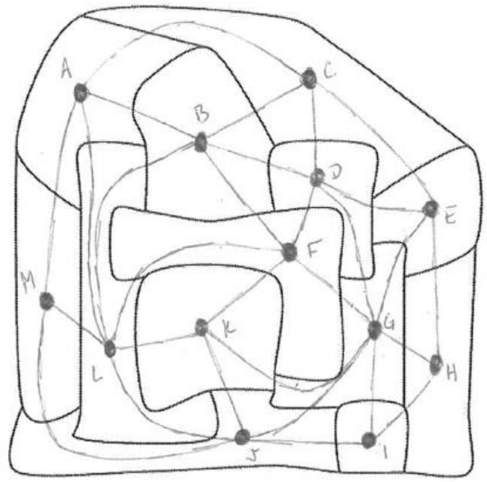
Perhaps the most famous graph theory problem is how to color maps.

Given any map of countries, states, counties, etc., how many colors are needed to color each region on the map so that neighboring regions are colored differently?

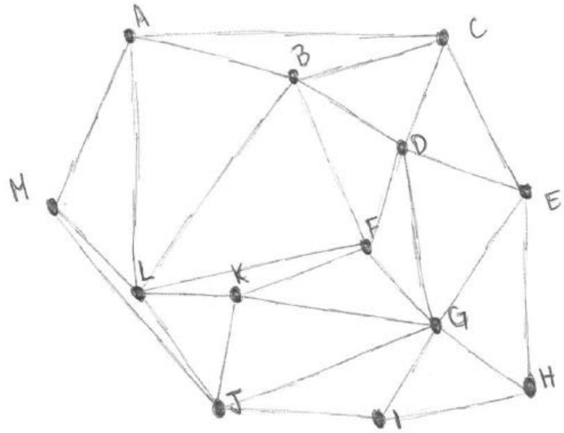
Actual map makers usually use around seven colors. For one thing, they require watery regions to be a specific color, and with a lot of colors it is easier to find a permissible coloring. We want to know whether there is a smaller palette that will work for any map.

DEFINITIONS

How is this related to graph theory? Well, if we place a vertex in the center of each region (say in the capital of each state) and then connect two vertices if their states share a border, we get a graph.



Coloring regions on the map corresponds to coloring the vertices of the graph. Since neighboring regions cannot be colored the same, our graph cannot have vertices colored the same when those vertices are adjacent.



In general, given any graph G,G, a coloring of the vertices is called (not surprisingly) a ***vertex coloring***.

If the vertex coloring has the property that adjacent vertices are colored differently, then the coloring is called ***proper***. Every graph has a proper vertex coloring. For example, you could color every vertex with a different color. But often you can do better.

The smallest number of colors needed to get a proper vertex coloring is called the ***chromatic number*** of the graph.

CLUB EXAMPLE

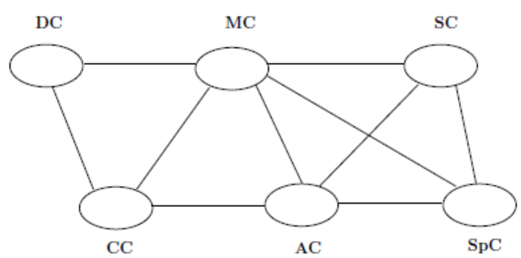
The coloring problem is not limited to helping us with maps. Suppose you were given the responsibility to schedule the meeting times of all the clubs in your high School. The first problem is that some students belong to more than one club, so not all of the clubs can meet on the same day. Secondly, the school does not want to be open every day for after school clubs. Thus you need to schedule as few days of the week for the clubs as possible. Below is the list of clubs and club members who belong to more than one club.

Clubs and Members

Clubs	Students in Multi-Clubs
Math Club	Dayna, Dale, Kristy
Debate Club	Kristy, Dayna, Travis
Science Club	Dale
Computer Club	Kristy, Rachel, Travis
Art Club	Rachel, Dale
Spanish Club	Dale

What is the minimum number of days needed so that no two clubs sharing a member meet on the same day?

When making a graph for the map, we connected countries that could not be colored the same color. Likewise, we want to connect clubs that cannot be scheduled together.



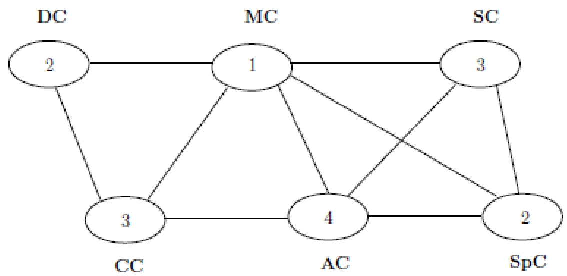
Now, say we schedule Math Club on the first day. Then nothing else can be scheduled on the first day.

Next, let's schedule Debate Club on the second day. Computer Club cannot be the same as either Math or Debate, so schedule it on the third day.

Science Club needs to be different from Math Club, not not from Computer Club, so make it the third day.

Spanish Club needs to be different from Science and Math, but not Debate, so make it the second day.

Art Club needs to be different from Computer, Math, Science, and Spanish, so we will need a fourth day.



Thus, we can schedule the following

Day 1	Day 2	Day 3	Day 4
Math Club	Debate Club Spanish Club	Science Club Computer Club	Art Club

Since it took four days to schedule all the clubs, we would say that the *chromatic number* of the graph is 4.

NOTE:

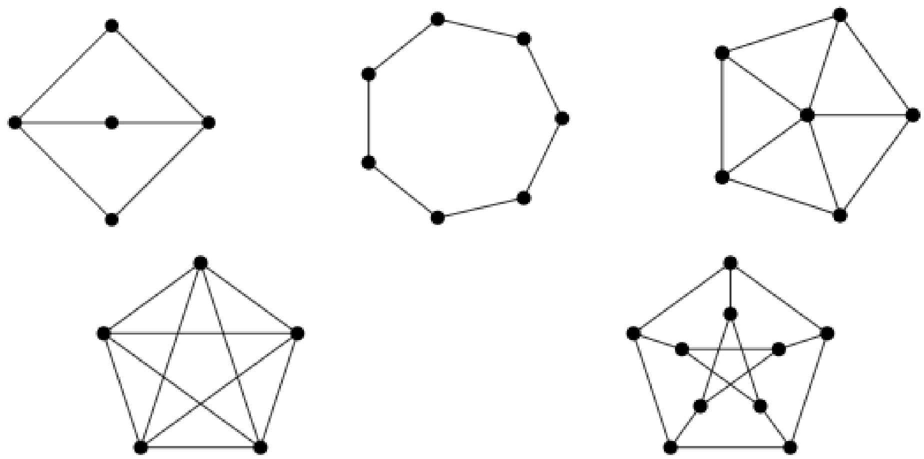
As you go through the following problem set, you will be asked to solve many similar problems, finding the minimum number of colors, days, closets, etc. There is not an easy formula or method for 'solving' a graph in this way. As you attack the problems, consider a few questions:

- Do you find yourself settling on any methods of attacking each problem?
- Are you seeing patterns, etc. that help you stay organized as you color?
- How confident you that you have really found the minimum each time?

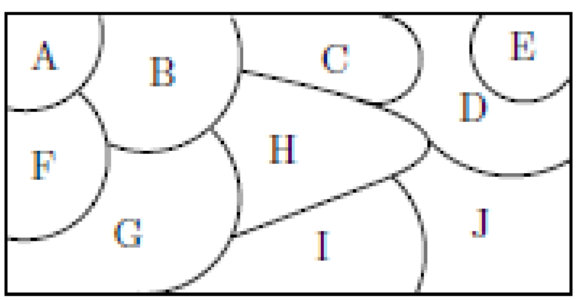
You will be asked to reflect on this at the end.

PROBLEM SET 3

1. Find the chromatic number for each of the graphs below.



2. Color the following map using only three colors.



3. Below is a list of chemicals together with a list of other chemicals with which each cannot be stored.

Chemicals	Cannot Be Stored With
1	2,5,7
2	1,3,5,4
3	2,4,6
4	2,3,7
5	1,2,6,7
6	5,3
7	1,4,5

How many different storage facilities are necessary in order to keep all seven chemicals?

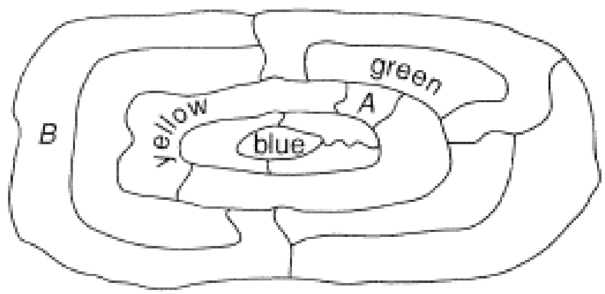
4. A local zoo wants to take visitors on animal feeding tours. They offer the following

   Tours:
   Tour 1: Visit lions, elephants, and giraffes
   Tour 2: Visit monkeys, hippos, and flamingos.
   Tour 3: Visit elephants, flamingos, and bears.
   Tour 4: Visit hippos, reptiles, and bears.
   Tour 5: Visit kangaroos, monkeys, and reptiles.

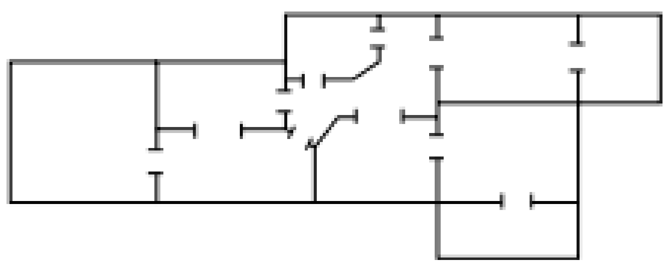
   The animals should not be fed more than once a day. Also, there is only room for one tour group at a time at any one site. Can these tours be scheduled using only Monday, Wednesday, and Friday?

5. Draw graphs to represent the maps below. Color the graphs and find the minimum number of colors needed to color each map.

6. Consider the map below, in which the countries are to be colored red, blue, green, and yellow. Show that country A must be red. What color is country B?

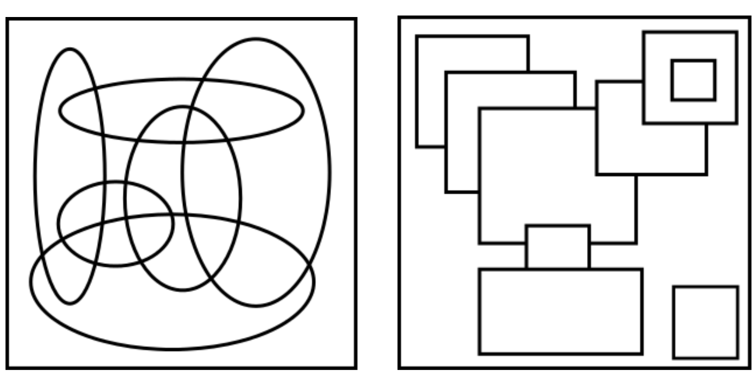


7. Oh no! Baby Euler has gotten into the hand paints. His favorite colors are blue and yellow. Baby Euler wants to paint each room in the house (including the hall) either blue or yellow such that every time he walks from one from to an adjacent room, the color changes. Is this possible?



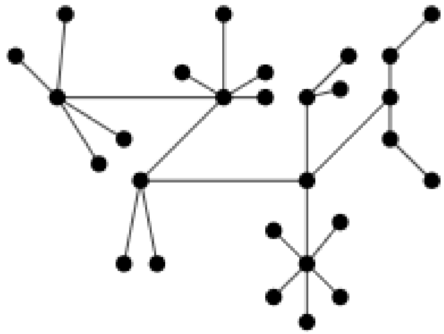
8. You have a set of magnetic alphabet letters (one of each of the 26 letters in the alphabet) that you need to put into boxes. For obvious reasons, you don't want to put two consecutive letters in the same box. What is the fewest number of boxes you need (assuming the boxes are able to hold as many letters as they need to)?

9. Consider these two images, one made from intersecting ellipses and one made from overlapping rectangles (include the outer square boundaries in the image)



What is the minimum number of colors needed for each, so no overlapping or adjacent regions share a color?

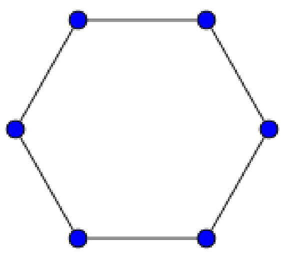
10. Remember that a tree is a connected graph with no cycles, like the following



a. Describe a procedure to color the tree above.
b. What is the chromatic number of the tree?
c. All trees should have the same chromatic number which is, hopefully, the number you found above. Explain why that make sense.

11. Describe how you would color a cyclic graph. What are the chromatic numbers of cyclic graphs?

Ex: Cyclic Graph C_6



12. Looking back, what can you say about the chromatic numbers of complete graphs and bipartite graphs?

COLORING ANALYSIS

Write your work, answers, and observations in your notebook.

Though graph coloring problems relate to far more applications than the map coloring problem, map coloring remains one of the most well-known and has a fascinating history.

Since the mid-1800s, mathematicians have believed the any map should be 4-colorable, but no one could prove it must be the case. In 1879, Alfred Kemp released a proof which mathematicians accepted for eleven years, when a mistake was found by Percy John Heawood. Heawood in turn used the failed proof to prove that all maps must be 5-colorable, but could not prove it for 4. Mathematicians have also tried to disprove the four color theorem – that is, to create a map such that five or more colors was needed. None have been successful at this either.

Today, the four color theorem is generally considered proven through a number of computer based proofs from 1976 to 2005. These proofs use a computer to check enough different cases that cover any imaginable configuration, thus they declare the theorem proven.

There is much more to this story, including the fact that mathematicians do not all agree on how much credence to give computer based proofs. It is worth researching if you are interested!

Back to graph theory, there are a number of algorithms (the greedy algorithm, distributed algorithms, etc.) which give direction about how to proceed through a graph and assign 'colors.' However, none of the algorithms *ensure* a minimum coloring.

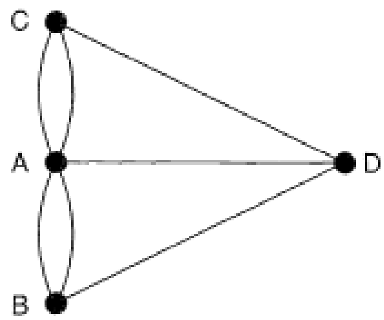
Now that you have completed the problems in Problem Set 3, reflect on the process of finding the chromatic number for a graph. Did you use a similar system each time to minimize the number of colors you used? Describe your system. Did you discover any patterns that helped you go more quickly in future problems? Etc.

SECTION IV: PATHS

INTRODUCTION

The problem of the seven bridges of Konigsberg is looking for a path across the bridges that crosses each bridge once and only once. If we rephrase the problem, just referring to the graph we created, we are looking for a path through this graph that travels each edge once and only once. Such a path through any graph we will call a Eulerian Path.

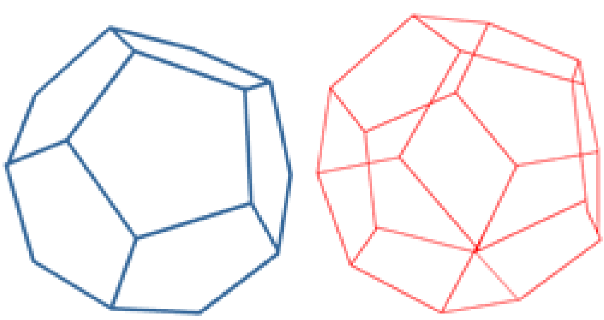
Eulerian Path: A path through a graph that travels each edge once and only once.



What if we wanted to change the question a bit and now want to try to travel to each land mass exactly once? In other words, what if we want a path through the graph that travels through each *vertex* once and only once? This is quite easy for the Konigsberg graph, but not for all graphs. This is called a Hamiltonian Path after another mathematician.

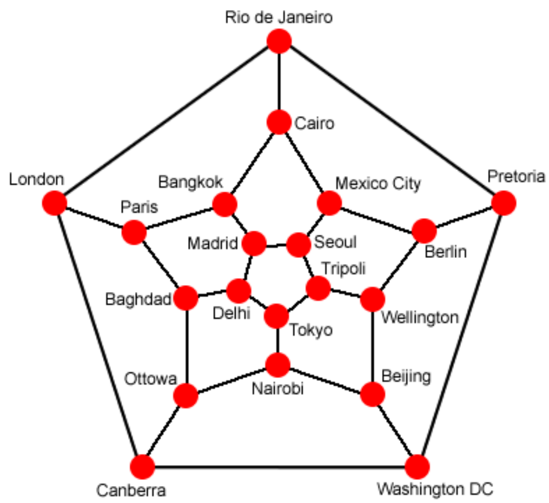
Hamiltonian Path: A path through a graph that travels each vertex once and only once.

In 1859, the Irish mathematician Sir William Rowan Hamilton devised a puzzle with a regular dodecahedron made of wood. Here is a dodecahedron:



He labelled each of the vertices with the name of an important city. The challenge was to find a route along the edges of the dodecahedron which visited every city exactly once and returned to the start.

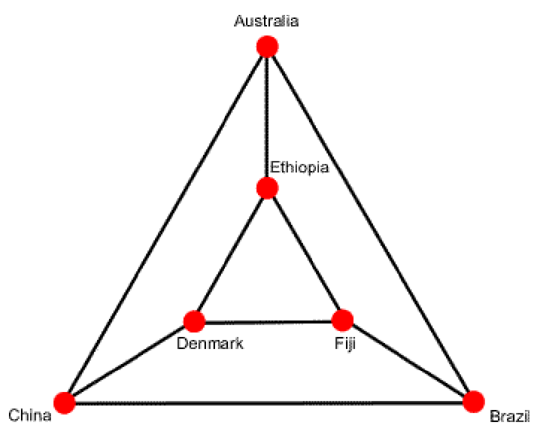
The dodecahedron is a convex polygon, so it can be drawn as a planar graph! Here is a graph which represents the dodecahedron. Can you see how each of the 20 vertices, 30 edges and 12 pentagonal faces is represented in the graph?



I start my journey in Rio de Janeiro and visit all the cities as Hamilton described, passing through Canberra before Madrid, and then returning to Rio. What route could I have taken?

Can you find any other ways of making this journey?

Here is a simpler network of countries:



How many different ways are there of visiting each of these countries once and only once, beginning and ending at Australia?

DEFINITIONS

Walk: A sequence of edges connecting one vertex to another

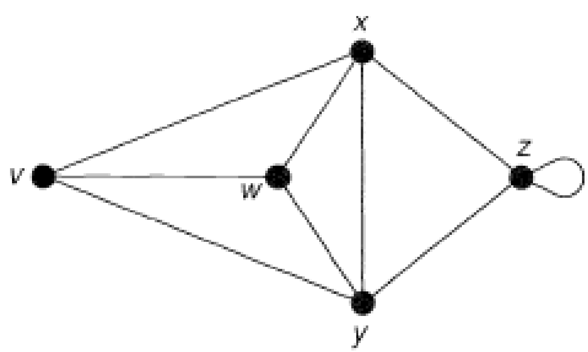
Trail: A walk in which no edge appears more than once

Path: A walk in which no vertex or edge appears more than once

Closed: A trail or path is closed if it ends at the vertex where it began.

Cycle: A closed path

Example:



v - w - x - z - y - w is an example of a trail.

v - w - x - z is an example of a path.

v - w - x - z - y - x to v is a closed trail.

v - w - y - v is a cycle.

Eulerian Walk: A walk that travels each edge of the graph once and only once.

Eulerian Cycle: A cycle that uses every edge of the graph once and only once.

Example

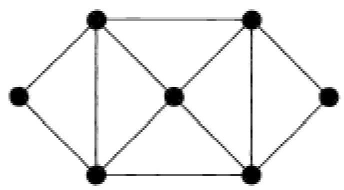


Fig. 6.1

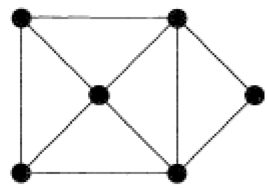


Fig. 6.2

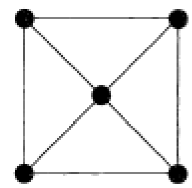


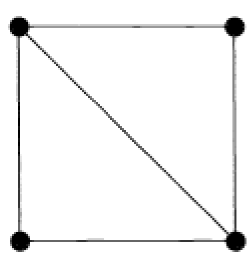
Fig. 6.3

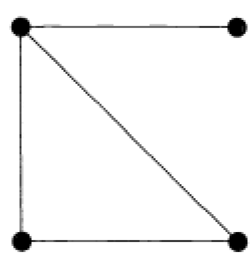
Fig 6.1 contains a Eulerian Cycle
Fig 6.2 contains a Eulerian Walk, but not Cycle
Fig 6.3 contains neither.

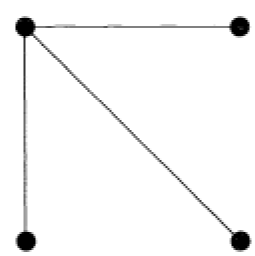
Hamiltonian Walk: A walk that travels each vertex of the graph once and only once.

Hamiltonian Cycle: A cycle that uses every vertex of a graph exactly once, (ends on the vertex started on)

Example



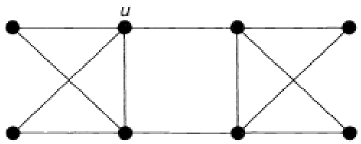




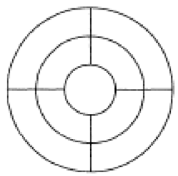
The first graph contains a Hamiltonian Cycle.
The second graph contains a Hamiltonian Walk, but not Cycle.
The third graph contains neither.

PROBLEM SET 4A

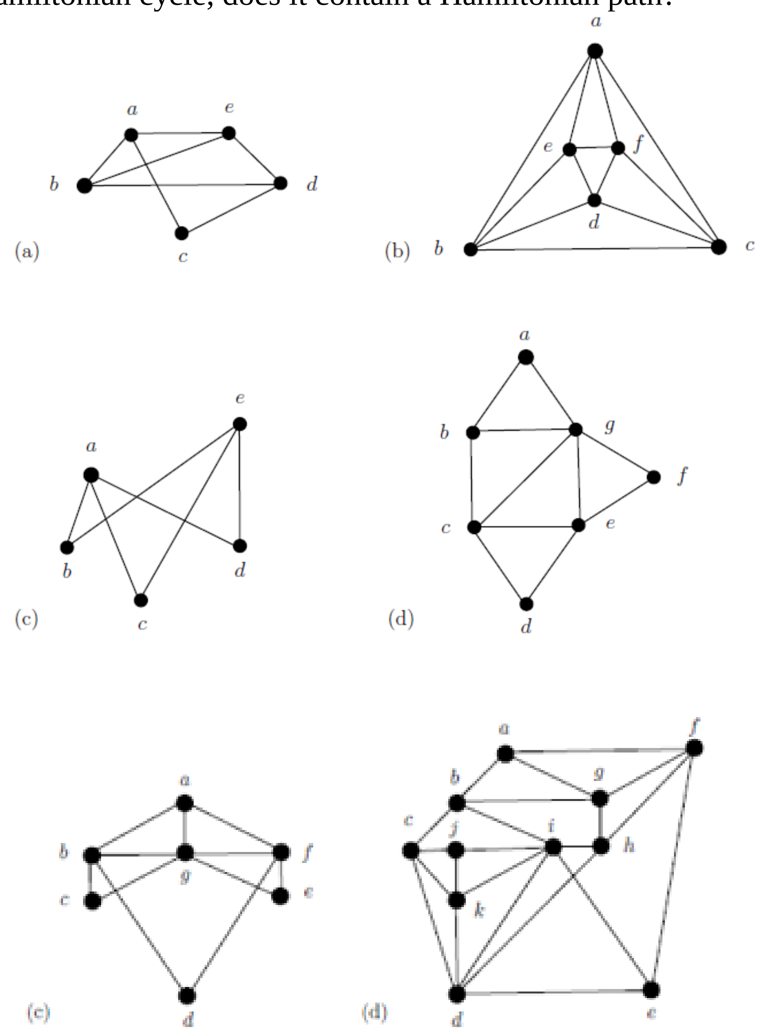
1. Find a Eulerian trail for the graph



2. How many continuous pen strokes are needed to draw the diagram below without repeating any line?



3. Determine if the following graphs contain a Hamiltonian cycle. If it does not contain a Hamiltonian cycle, does it contain a Hamiltonian path?



INVESTIGATION:

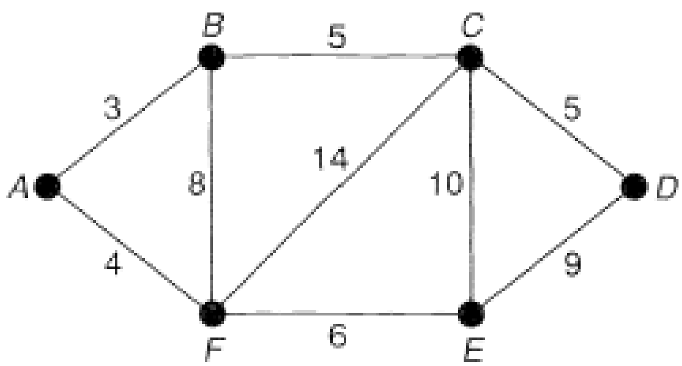
Write your work, answers, and observations in your notebook.

Chinese Postman Problem:

In this problem, discussed by the Chinese mathematician Mei-Ku Kwan, a postman wishes to deliver his letters, covering the least possible total distance and *returning to his starting point*. He must obviously traverse each road in his route at least once, but should avoid covering too many roads more than once.

This problem can be reformulated in terms of a *weighted* graph, where the graph corresponds to the network of roads, and the weight of each edge is the length of the corresponding road. In this reformulation, the requirement is to find a closed walk of minimum total weight that includes each edge (road) at least once.

(If the graph has a Eulerian cycle, this is obviously the best path, since no roads would be repeated.)

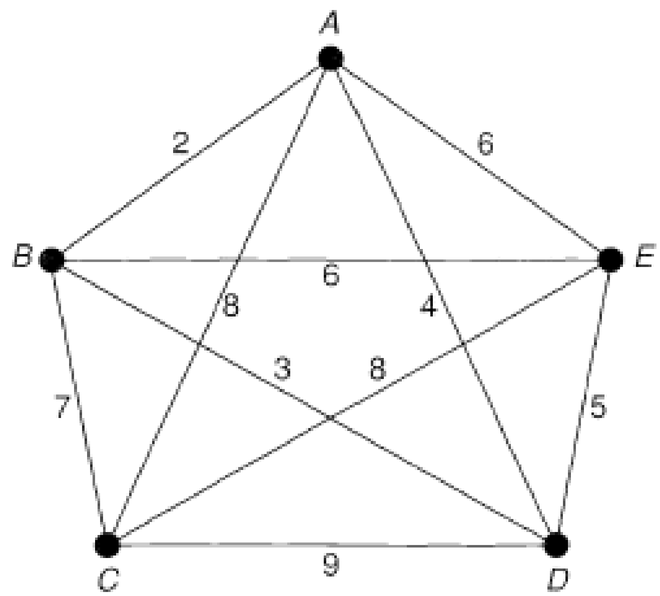


What is the best answer you can find for this problem?

Traveling Salesman Problem

In this problem, a traveling salesman wishes to visit several given cities and return to his starting point, covering the least possible total distance.

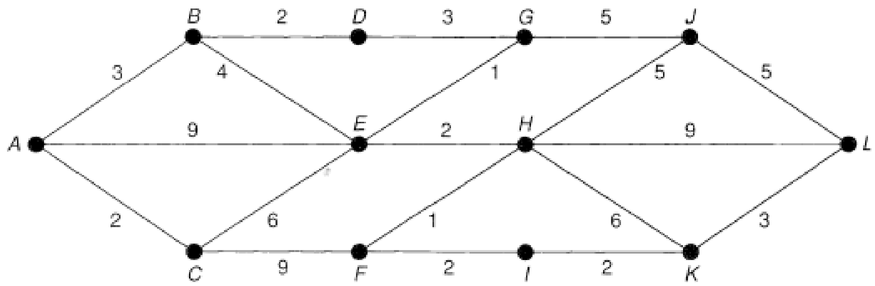
This problem is very close to Hamilton's problem with the dodecahedron a few pages ago. Like the Chinese Postman Problem, the Traveling Salesman Problem can also be reformulated in terms of weighted graphs. In this case, the requirement is to find a *Hamiltonian* cycle of least possible total weight in a weighted complete graph.



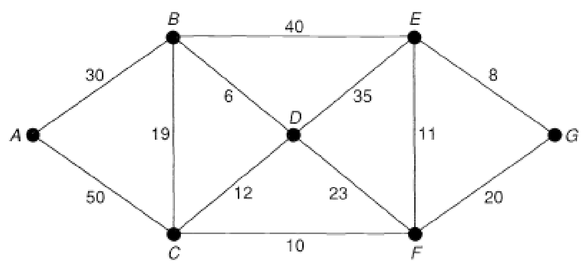
What is the best answer you can find to this problem?

PROBLEM SET 4B

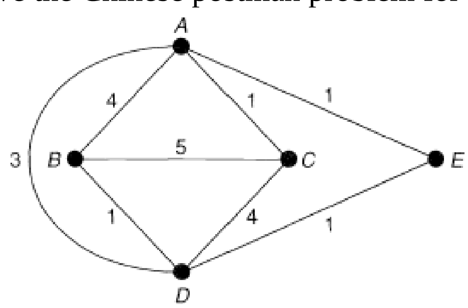
1. What is the length of the shortest path from A to L?



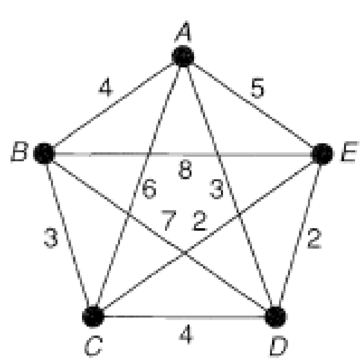
2. Find the shortest path from A to G on the graph below.



3. Solve the Chinese postman problem for the weighted graph below.



4. Solve the traveling salesman problem for the weighted graph below.



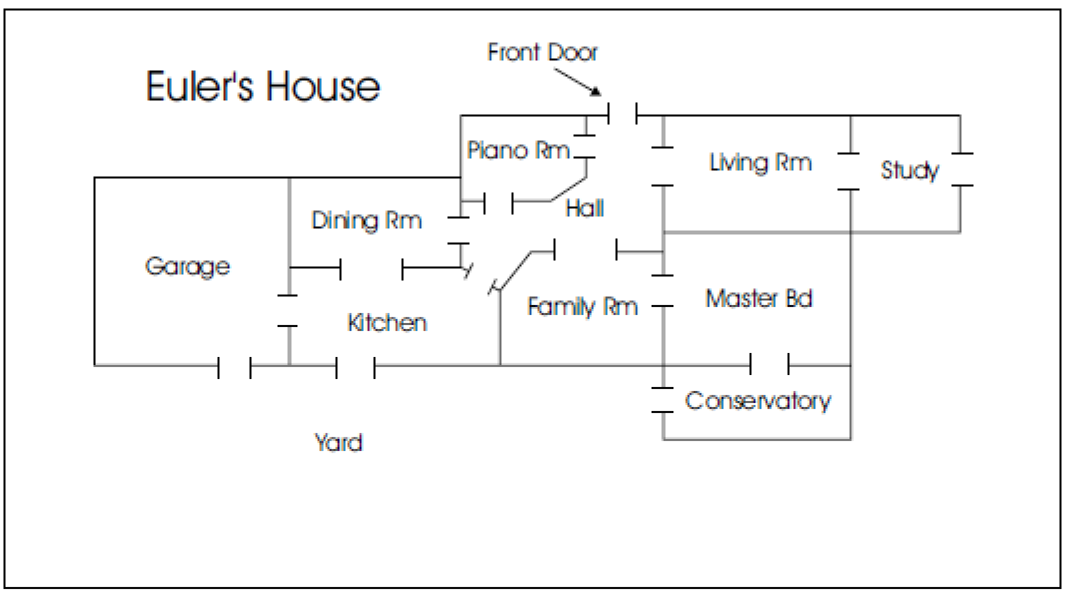
5. Suppose you are a salesperson who lives in Johnson City. You want to travel to several different cities, say Chicago, Atlanta, Washington D.C., exactly once and then return home to Johnson City. The list below represents the trips that are available to you and the costs of making a trip between the cities.

Cities	Cost
Atlanta - Chicago	$400
Atlanta - Johnson City	$550
Atlanta - Washington D.C.	$1,260
Chicago - Johnson City	$270
Chicago - Washington D.C.	$910
Johnson City - Washington D.C.	$670

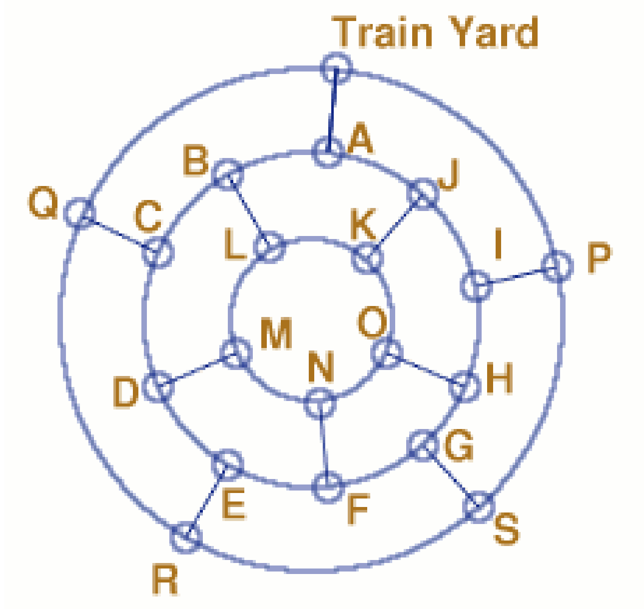
Since you own your business, it is important to use the least amount of money for your trip. To save money, try to find the least expensive route that begins in Johnson City, visits each of the other cities exactly once, and returns to Johnson City.

6. How many tours that visit each vertex once and only once can be traced along the edges of a cube? How many of these tours can return to the starting point thus completing a Hamiltonian Circuit?

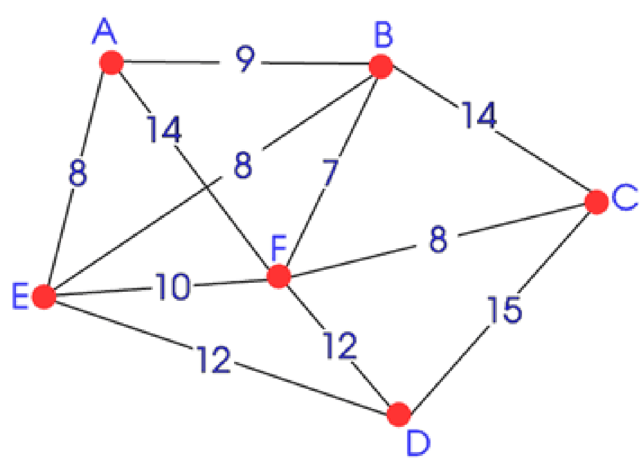
7. Baby Euler has just learned to walk. He is curious to know if he can walk through every doorway in his house exactly once, and return to the room he started in. Will baby euler succeed? Can baby Euler walk through every door exactly once and return to a different place than where he started? What if the front door is closed?



8. This drawing shows the train track joining the Train Yard to all the stations labelled from A to S. Find a way for a train to call at all the stations and return to the Yard.



9. The graph below represents a salesman's area of activity with shops at A, B, C, D, E and F. (It isn't drawn to scale.)



The numbers represent the distance in kilometres between the places that salesman must visit each day. What route around the shops has the minimum total distance?

PATH ANALYSIS

Write your work, answers, and observations in your notebook.

Like with coloring, there are some methods out there that mathematicians use for finding optimal paths. However, none of the methods are foolproof. They do not always work.

Now that you have completed the problems in Problem Set 4B, reflect on the process of finding optimal paths. Did you use a similar system each time to minimize travel or cost? Describe your system. Did you discover any patterns that helped you go more quickly in future problems? How often did you feel confident that you had actually found the optimal path? Etc.

SECTION V: INDEPENDENT EXPLORATION

In this section, you will get to choose from a variety of further explorations into graph theory. You will devise a plan for exploring, create explorations and problems, and keep track or your results.

Hopefully you have seen that the possible applications of graph theory are many. Graph theory is a new enough discipline that there are likely areas of application no one has yet considered or exhausted. Feel free to try something new if you have an idea of how you might use graphs!

Mathematical questions can spring both from the real world, but also the abstract world of mathematics. You could approach this final section through one or both of these lenses. Do you have ideas for different applications of graphs? Explore those! Do you have more abstract questions like those related to Euler's theorem or other patterns we saw with numbers of edges, vertices, etc.? Explore those! There are many abstract patterns and theorems in graph theory that we did not even touch here.

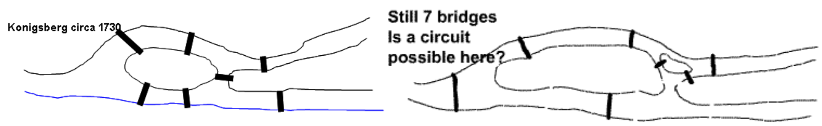
The following pages list a number of possibilities for what you might explore in this section. Most of the ideas involve a concrete application. Feel free to choose any of these, but also feel free to look at your own idea.

Either way, take some real time to make a plan for how you want to approach your topic or question. What specific questions are you trying to answer? Will you make up a number of concrete examples to do and then look for patterns? will you try to make some definitions? Will you make up an exploration to help you hone in on a pattern?

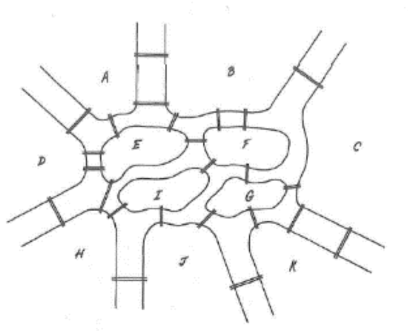
Once you do the exploring, leave a little time to think through what you have accomplished and draw some conclusions. Did you answer the questions you started with? If not, what did you achieve? Even if your conclusion is simply that you now know all the things you tried didn't work, say that!

EXPLORATION IDEAS

1. Konigsberg Plus:
   Expand on Euler's Konigsberg Bridge Problem. While there is no solution for the original set of bridges: If we add an island and adjust the bridges just slightly: It is possible!

   Another Ex: Can you take a walk and cross each bridge below exactly once? If so, where do you start and finish? Blow up the bridge from H to I and answer the same two questions.

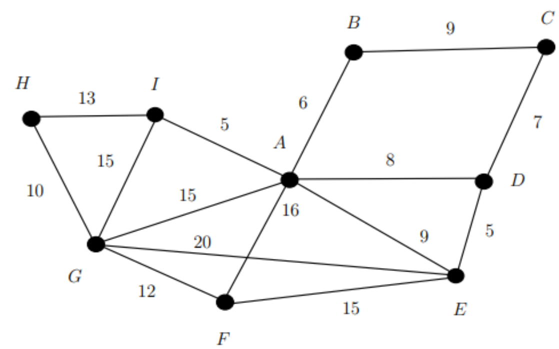
   Now experiment with different numbers of islands and bridges. What patterns can you find? Are there certain numbers of bridges or land masses that work/don't work?

2. Chess
   There are so many questions one could ask about a chess board and the game of chess that graph theory could help us think through and solve. For example:
   a. How many pieces can traverse the board, visiting (landing on, not just crossing over) each square once and only once?
   b. When a knight can traverse a chess board as in part a, it is called a knight's tour. Experiment with different size boards: what board dimensions allow a knight to make a tour. Which do not? Are there any patterns you can find?
   c. "Domination" of a chess board occurs when your pieces are set up such that any square can be attacked by a piece. Is it possible to dominate a chess board with the allotted pieces? What would be the minimum number? Experiment with pieces not allowed (i.e. what is the minimum number of queens you would need to dominate a chess board?) Experiment with different size boards.

   Explore these or other problems you can think of relating to the game of chess.
   In order to work on this problem, you need to know how chess works. If you need to look up the rules of chess, how pieces are allowed to move, the size of the board, etc. you may.

3. Infinite Graphs
   The key for anything to be infinite is that there needs to be a pattern such that, even though you cannot write or draw the entire thing, you always know what comes next What would it take for a graph to be infinite? Can you imagine one? Draw one? Write a definition? What might an infinite graph be used to describe? Can you think of some application problems? Can you solve them?

4. Trees
   Remember a tree is a graph containing no cycles. We touched on the tree a few times throughout the problem. What else could you explore about trees? Can you find any interesting properties? What kind of applications might you use trees to help you solve? One kind of application you might investigate involves "minimum spanning trees." For example, say you live in a county with a number of cities (A-I), connected by roads with mileage between the cities as shown below. You are responsible for deciding the minimum amount of road that needs to be plowed in a snowstorm such that someone could get from any city to any other city, even if by an indirect route.



   Explain difference between this kind of problem and an Euler or Hamilton walk. Make up some examples. Can you find some good methods for solving such a problem? Are there other applications besides roads?

5. Tournaments
   How can you use graph theory to help in the setting up and evaluating of tournaments? How is a typical bracket tournament an example of a graph? Say you have a round robin tournament between a number of teams, where every team plays every other team. How will you rank the teams at the end of the tournament? What if you have a tournament where each team plays a set number of games against randomly chosen opponents, but each team does not play every other team. How might you rank the teams at the end? One Hint: consider using arrows instead of edges in your graph.

6. If you have another idea for an exploration, great! Write out an explanation of your idea, what you want to explore or what questions you want to answer, and how you are going to go about it.